Shallow water approximations for water waves over a moving bottom (Mathematical Analysis in Fluid and Gas Dynamics)

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1 Introduction

In this communication we are concerned with model equations for generation and propagation of tsunamis. In a standard tsunami model, the shallow water equations

$$\begin{cases} \eta_t + \nabla\cdot\big((h+\eta-b_1)u\big) = 0, \\ u_t + (u\cdot\nabla)u + g\nabla\eta = 0 \end{cases}$$

are used to simulate the propagation of tsunami under the assumption that the initial profile of water surface is equal to the permanent shift of the seabed and the initial velocity field is zero, that is,

$$\eta = b_1 - b_0, \quad u = 0 \quad \text{at} \quad t = 0,$$

where η is the elevation of the water surface, u is the velocity field in the horizontal direction on the water surface, h is the mean depth of the water, g is the gravitational constant, b_0 is the bottom topography before the submarine earthquake, and b_1 is that after the earthquake. In fact, in [6] it was shown that the solution of the full water wave problem can be approximated by the solution of this tsunami model in the scaling regime $\delta^2 \ll \varepsilon \ll 1$ under appropriate assumptions on the initial data and the bottom topography. Here the non-dimensional parameters δ and ε are defined by

$$\delta = \frac{h}{\lambda}, \qquad \varepsilon = \frac{t_0}{\lambda/\sqrt{gh}},$$

where λ is a typical wave length and t_0 is the time when the submarine earthquake takes place. We note that $\sqrt{gh}$ is the propagation speed of the linear shallow water waves, so that $\lambda/\sqrt{gh}$ is a time period of the waves. It is natural to assume the condition $\delta^2 \ll \varepsilon \ll 1$, since tsunamis have very long wavelength and very long time period.

However, very rarely, the condition $\delta^2 \ll \varepsilon \ll 1$ is not satisfied, particularly, the condition on ε. One of such events is the Meiji-Sanriku earthquake, which occurred at

June 15 in 1896. The seismic scale of this earthquake was small, but it continued for several minutes. As a result, huge tsunami attacked the Sanriku coast line. To simulate such a tsunami, it might be better to consider the limit $\delta \to 0$ keeping ε is of order one. In this communication we will consider this kind of tsunamis, so that in the following we always assume that $\varepsilon = 1$. In this case, the standard tsunami model should be replaced by the shallow water equations with a source term

$$\begin{cases} \eta_t + \nabla \cdot \big((h + \eta - b)u\big) = b_t, \\ u_t + (u \cdot \nabla)u + g\nabla\eta = 0 \end{cases}$$

with zero initial conditions, where b is the bottom topography. In fact, using the techniques in [6] we can show that the solution of the full water wave problem can be approximated by the solution of the above tsunami model in the scaling regime $\delta \ll 1$ and $\varepsilon = 1$. Therefore, in this communication we will consider a higher order approximation.

It was shown by Li [10] that the solution of the two-dimensional water waves over a flat bottom can be approximated by the solution of the so-called Green–Nagdhi equations

$$\begin{cases} \eta_t + \big((1 + \eta)u\big)_x = 0, \\ u_t + uu_x + \eta_x = \frac{1}{3}\delta^2(1+\eta)^{-1}\big((1+\eta)^3(u_{xt} + uu_{xx} - u_x^2)\big)_x \end{cases}$$

up to order $O(\delta^4)$. In a dimensional form the Green–Nagdhi equations are written by

$$\begin{cases} \eta_t + \big((h + \eta)u\big)_x = 0, \\ u_t + uu_x + g\eta_x = \frac{1}{3}(h+\eta)^{-1}\big((h+\eta)^3(u_{xt} + uu_{xx} - u_x^2)\big)_x. \end{cases}$$

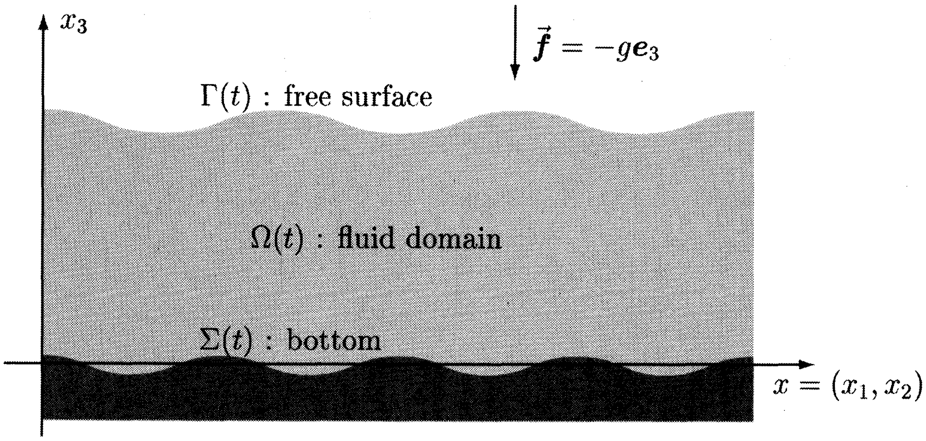
Alvarez-Samaniego and Lannes [1] extended her result to the three-dimensional water waves over a nonflat bottom by using the Nash–Moser technique to show the existence of solution, so that they imposed much regularity of the initial data. In this communication, we extend the result to the case of moving bottom without using the Nash–Moser technique. Therefore, in our result the regularity assumption on the initial data is much weaker than those in [1].

2 Formulation of the Problem

We proceed to formulate the problem precisely. Let $x = (x_1, x_2)$ be the horizontal spatial variables and x_3 the vertical spatial variable. We denote by $X = (x, x_3) = (x_1, x_2, x_3)$ the whole spatial variables. We will consider a water wave in three dimensional space and assume that the domain $\Omega(t)$ occupied by the water at time t, the water surface $\Gamma(t)$, and the bottom $\Sigma(t)$ are of the forms

$$\begin{aligned} \Omega(t) &= \big\{X = (x, x_3) \in \mathbf{R}^3\,;\, b(x,t) < x_3 < h + \eta(x,t)\big\}, \\ \Gamma(t) &= \big\{X = (x, x_3) \in \mathbf{R}^3\,;\, x_3 = h + \eta(x,t)\big\}, \\ \Sigma(t) &= \big\{X = (x, x_3) \in \mathbf{R}^3\,;\, x_3 = b(x,t)\big\}, \end{aligned}$$

where h is the mean depth of the water. The shape of the fluid region is shown in the following illustration.



The functions b and η represent the bottom topography and the surface elevation, respectively. It is very important to predict the deformation process of the seabed, so that we have to analyze the behavior of this function b. However, in this communication we assume that b is a given function and we concentrate our attention on analyzing the behavior of the function η, namely, the water surface.

We assume that the water is incompressible and inviscid fluid, and that the flow is irrotational. Then, the motion of the water is described by the velocity potential $\Phi = \Phi(X, t)$ satisfying the equation

$$\Delta_X \Phi = 0 \quad \text{in} \quad \Omega(t), \tag{2.1}$$

where Δ_X is the Laplacian with respect to X, that is, $\Delta_X = \Delta + \partial_3^2$ and $\Delta = \partial_1^2 + \partial_2^2$. The boundary conditions on the water surface are given by

$$\begin{cases} \eta_t + \nabla\Phi \cdot \nabla\eta - \partial_3\Phi = 0, \\ \Phi_t + \frac{1}{2}|\nabla_X \Phi|^2 + g\eta = 0 \quad \text{on} \quad \Gamma(t), \end{cases} \tag{2.2}$$

where $\nabla = (\partial_1, \partial_2)^T$ and $\nabla_X = (\partial_1, \partial_2, \partial_3)^T$ are the gradients with respect to $x = (x_1, x_2)$ and to $X = (x, x_3)$, respectively, and g is the gravitational constant. The first equation is the kinematical condition and the second one is the restriction of Bernoulli's law on the water surface. The kinematical boundary condition on the bottom is given by

$$b_t + \nabla\Phi \cdot \nabla b - \partial_3\Phi = 0 \quad \text{on} \quad \Sigma(t). \tag{2.3}$$

Finally, we impose the initial conditions

$$\eta = \eta_0, \quad \Phi = \Phi_0 \quad \text{at} \quad t = 0. \tag{2.4}$$

These are the basic equations for the full water wave problem.

Next, we rewrite the equations (2.1)–(2.3) in an appropriate non-dimensional form. Let λ be the typical wave length and h the mean depth. We introduce a non-dimensional parameter δ by $\delta = h/\lambda$ and rescale the independent and dependent variables by

$$x = \lambda\tilde{x}, \quad x_3 = h\tilde{x}_3, \quad t = \frac{\lambda}{\sqrt{gh}}\tilde{t}, \quad \Phi = \lambda\sqrt{gh}\tilde{\Phi}, \quad \eta = h\tilde{\eta}, \quad b = h\tilde{b}. \tag{2.5}$$

Putting these into (2.1)–(2.3) and dropping the tilde sign in the notation we obtain

$$\delta^2\Delta\Phi + \partial_3^2\Phi = 0 \quad \text{in} \quad \Omega(t), \tag{2.6}$$

$$\begin{cases} \delta^2(\eta_t + \nabla\Phi\cdot\nabla\eta) - \partial_3\Phi = 0, \\ \delta^2\big(\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + \eta\big) + \frac{1}{2}(\partial_3\Phi)^2 = 0 \quad \text{on} \quad \Gamma(t), \end{cases} \tag{2.7}$$

$$\delta^2(b_t + \nabla\Phi\cdot\nabla b) - \partial_3\Phi = 0 \quad \text{on} \quad \Sigma(t), \tag{2.8}$$

where

$$\begin{aligned} \Omega(t) &= \big\{X = (x, x_3) \in \mathbf{R}^3\,;\, b(x,t) < x_3 < 1 + \eta(x,t)\big\}, \\ \Gamma(t) &= \big\{X = (x, x_3) \in \mathbf{R}^3\,;\, x_3 = 1 + \eta(x,t)\big\}, \\ \Sigma(t) &= \big\{X = (x, x_3) \in \mathbf{R}^3\,;\, x_3 = b(x,t)\big\}. \end{aligned}$$

Since we are interested in asymptotic behavior of the solution when $\delta \to +0$, we always assume $0 < \delta \le 1$ in the following.

As in the usual way, we transform equivalently the initial value problem (2.6)–(2.8) and (2.4) to a problem on the water surface. To this end, we introduce a Dirichlet-to-Neumann map Λ^{DN} and a Neumann-to-Neumann map Λ^{NN} in the following way. In the definition the time t is arbitrarily fixed, so that we omit to write the dependence of t.

Definition 2.1 Under appropriate assumptions on η and b, for any functions ϕ on the water surface Γ and β on the seabed Σ in some classes, the boundary value problem

$$\begin{cases} \Delta\Phi + \delta^{-2}\partial_3^2\Phi = 0 & \text{in} \quad \Omega, \\ \Phi = \phi & \text{on} \quad \Gamma, \\ \delta^{-2}\partial_3\Phi - \nabla b\cdot\nabla\Phi = \beta & \text{on} \quad \Sigma \end{cases} \tag{2.9}$$

has a unique solution Φ. Using the solution we define $\Lambda^{\text{DN}}(\eta, b, \delta)$ and $\Lambda^{\text{NN}}(\eta, b, \delta)$ by

$$\begin{aligned} \Lambda^{\text{DN}}(\eta, b, \delta)\phi + \Lambda^{\text{NN}}(\eta, b, \delta)\beta &= \delta^{-2}(\partial_3\Phi)(\cdot, 1 + \eta(\cdot)) - \nabla\eta\cdot(\nabla\Phi)(\cdot, 1+\eta(\cdot)) \\ &= (\delta^{-2}\partial_3\Phi - \nabla\eta\cdot\nabla\Phi)|_\Gamma. \end{aligned} \tag{2.10}$$

The solution Φ will be denoted by $(\phi, \beta)^h$.

We should remark that both of the maps $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta, b, \delta)$ and $\Lambda^{\mathrm{NN}} = \Lambda^{\mathrm{NN}}(\eta, b, \delta)$ are linear operators acting on ϕ and β, respectively. However, they depend also on the unknown function η and the dependence on η is strongly nonlinear.

Now, we introduce a new unknown function ϕ by

$$\phi(x,t) = \Phi(x, 1+\eta(x,t), t) = \Phi|_{\Gamma(t)}, \tag{2.11}$$

which is the trace of the velocity potential on the water surface. Then, it holds that

$$\begin{cases} \phi_t = (\Phi_t + (\partial_3\Phi)\eta_t)|_{\Gamma(t)}, \\ \nabla\phi = (\nabla\Phi + (\partial_3\Phi)\nabla\eta)|_{\Gamma(t)}. \end{cases} \tag{2.12}$$

On the other hand, it follows from (2.6), (2.8), and (2.11) that Φ satisfies the boundary value problem (2.9) with β replaced by b_t, so that we have

$$\Lambda^{\mathrm{DN}}\phi + \Lambda^{\mathrm{NN}}b_t = (\delta^{-2}\partial_3\Phi - \nabla\eta\cdot\nabla\Phi)|_{\Gamma(t)}. \tag{2.13}$$

These relations (2.12) and (2.13) imply that

$$\begin{cases} (\partial_3\Phi)|_{\Gamma(t)} = \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}\phi + \Lambda^{\mathrm{NN}}b_t + \nabla\eta\cdot\nabla\phi), \\ (\nabla\Phi)|_{\Gamma(t)} = \nabla\phi - \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}\phi + \Lambda^{\mathrm{NN}}b_t + \nabla\eta\cdot\nabla\phi)\nabla\eta, \\ \Phi_t|_{\Gamma(t)} = \phi_t - \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}\phi + \Lambda^{\mathrm{NN}}b_t + \nabla\eta\cdot\nabla\phi)\eta_t. \end{cases}$$

Putting these into (2.7) we see that η and ϕ satisfy the following initial value problem.

$$\begin{cases} \eta_t - \Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi - \Lambda^{\mathrm{NN}}(\eta,b,\delta)b_t = 0, \\ \phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 \\ \quad -\frac{1}{2}\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}\big(\Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi + \Lambda^{\mathrm{NN}}(\eta,b,\delta)b_t + \nabla\eta\cdot\nabla\phi\big)^2 = 0, \end{cases} \tag{2.14}$$

$$\eta = \eta_0, \quad \phi = \phi_0 \quad \text{at} \quad t = 0, \tag{2.15}$$

where the initial datum ϕ_0 is determined by $\phi_0 = \Phi_0(\cdot, 1+\eta_0(\cdot))$. We will investigate this initial value problem (2.14) and (2.15) mathematically rigorously in this communication.

The following theorem is one of the main results in this paper and asserts the existence of the solution of (2.14) and (2.15) with uniform bounds of the solution on a time interval independent of small $\delta > 0$.

Theorem 2.1 *Let $s > 3$ and $M_0, c_0 > 0$. Then, there exist a time $T > 0$ and constants $C_0, \delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, $\eta_0 \in H^{s+7/2}$, $\nabla\phi_0 \in H^{s+3}$, and $b \in C([0,T]; H^{s+4})$ satisfying*

$$\begin{cases} \|b(t)\|_{s+4} + \|b_t(t)\|_{s+3} + \|b_{tt}(t)\|_{s+1} + \|b_{ttt}(t)\|_s \le M_0, \\ \|\eta_0\|_{s+7/2} + \|\nabla\phi_0\|_{s+3} \le M_0, \\ 1 + \eta_0(x) - b_0(x) \ge c_0 \quad \text{for} \quad (x,t) \in \mathbf{R}^2 \times [0,T], \end{cases}$$

the initial value problem (2.14) *and* (2.15) *has a unique solution* $(\eta,\phi)=(\eta^\delta,\phi^\delta)$ *on the time interval* [0,T] *satisfying*

$$\begin{cases} \|\eta^\delta(t)\|_{s+3}+\|\nabla\phi^\delta(t)\|_{s+2}+\|(\eta_t^\delta(t),\phi_t^\delta(t))\|_{s+2}\le C_0, \\ 1+\eta^\delta(x,t)-b(x,t)\ge c_0/2 \quad \text{for} \quad (x,t)\in\mathbf{R}^2\times[0,T], \quad \delta\in(0,\delta_0]. \end{cases}$$

3 Shallow water approximations

We proceed to study formally asymptotic behavior of the solution $(\eta^\delta,\phi^\delta)$ to the initial value problem (2.14) and (2.15) when $\delta\to+0$ and derive the shallow water equations and the Green–Nagdhi equations whose solutions approximate $(\eta^\delta,\phi^\delta)$ in a suitable sense.

In order to derive approximate equations to (2.14) we need to expand the Dirichlet-to-Neumann map $\Lambda^{\mathrm{DN}}=\Lambda^{\mathrm{DN}}(\eta,b,\delta)$ with respect to δ^2. Let Φ be the solution of the boundary value problem

$$\begin{cases} \Delta\Phi+\delta^{-2}\partial_3^2\Phi=0 & \text{in} \quad \Omega, \\ \Phi=\phi & \text{on} \quad \Gamma, \\ \delta^{-2}\partial_3\Phi-\nabla b\cdot\nabla\Phi=0 & \text{on} \quad \Sigma. \end{cases} \tag{3.1}$$

Here and in what follows, for simplicity we omit to write the dependence of the time t in the notation. By the first and the third equations in (3.1),

$$\begin{aligned}(\partial_3\Phi)(x,x_3)&=(\partial_3\Phi)(x,b(x))+\int_{b(x)}^{x_3}(\partial_3^2\Phi)(x,z)\mathrm{d}z \\ &=\delta^2\nabla b(x)\cdot\nabla\Phi(x,b(x))-\delta^2\int_{b(x)}^{x_3}(\Delta\Phi)(x,z)\mathrm{d}z,\end{aligned} \tag{3.2}$$

which implies that $(\partial_3\Phi)(X)=O(\delta^2)$. This and the second equation in (3.1) give

$$\begin{aligned}\Phi(x,x_3)&=\Phi(x,1+\eta(x))+\int_{1+\eta(x)}^{x_3}(\partial_3\Phi)(x,z)\mathrm{d}z \\ &=\phi(x)+O(\delta^2).\end{aligned} \tag{3.3}$$

Putting this into (3.2) yields that

$$\begin{aligned}(\partial_3\Phi)(x,x_3)&=\delta^2\nabla b(x)\cdot\nabla\phi(x)-\delta^2\int_{b(x)}^{x_3}\Delta\phi(x)\mathrm{d}z+O(\delta^4) \\ &=\delta^2\nabla b(x)\cdot\nabla\phi(x)-\delta^2\big(x_3-b(x)\big)\Delta\phi(x)+O(\delta^4).\end{aligned} \tag{3.4}$$

Hence, by the definition (2.10) with $\beta=0$ we have

$$\big(\Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi\big)(x)=-\nabla\cdot\big((1+\eta(x)-b(x))\nabla\phi(x)\big)+O(\delta^2). \tag{3.5}$$

We proceed to derive a higher order expansion of $\Lambda^{\mathrm{DN}}(\eta,b,\delta)$ up to order $O(\delta^4)$. Putting (3.4) into (3.3) we have

$$\begin{aligned}\Phi(x,x_3) =&\phi(x)+\delta^2\big(x_3-(1+\eta(x))\big)\nabla b(x)\cdot\nabla\phi(x)\\ &-\delta^2\big\{\tfrac{1}{2}\big(x_3^2-(1+\eta(x))^2\big)-\big(x_3-(1+\eta(x))\big)b(x)\big\}\Delta\phi(x)+O(\delta^4),\end{aligned}$$

which together with (3.2) implies that

$$\begin{aligned}&\partial_3\Phi(x,x_3)\\ &=\delta^2\big\{\nabla b(x)\cdot\nabla\phi(x)-(x_3-b(x))\Delta\phi(x)\big\}\\ &\quad+\delta^4\nabla b(x)\cdot\big\{\tfrac{1}{2}(1+\eta(x)-b(x))^2\nabla\Delta\phi(x)+(1+\eta(x)-b(x))(\nabla\eta(x)-\nabla b(x))\Delta\phi(x)\\ &\qquad-(1+\eta(x)-b(x))\nabla(\nabla b(x)\cdot\nabla\phi(x))-\nabla\eta(x)(\nabla b(x)\cdot\nabla\phi(x))\big\}\\ &\quad+\delta^4\big\{\big(\nabla\eta(x)(\nabla b(x)\cdot\nabla\phi(x))+2\nabla\eta(x)\cdot\nabla(\nabla b(x)\cdot\nabla\phi(x))\big)(x_3-b(x))\\ &\qquad+\tfrac{1}{2}(2+\eta(x)-x_3-b(x))(x_3-b(x))\Delta(\nabla b(x)\cdot\nabla\phi(x))\big\}\\ &\quad+\delta^4\big\{\big(2\nabla\eta(x)\cdot\nabla b(x)-|\nabla\eta(x)|^2-(1+\eta(x)-b(x))\Delta\eta(x)\big)(x_3-b(x))\big\}\\ &\qquad+\tfrac{1}{2}(2+\eta(x)-x_3-b(x))(x_3-b(x))\Delta b(x)\big\}\Delta\phi(x)\\ &\quad+\delta^4\big\{(2+\eta(x)-x_3-b(x))(x_3-b(x))\nabla b(x)\\ &\qquad-2(1+\eta(x)-b(x))(x_3-b(x))\nabla\eta(x)\big\}\cdot\nabla\Delta\phi(x)\\ &\quad+\delta^4\big\{\tfrac{1}{2}(2+\eta(x)-x_3-b(x))(x_3-b(x))b(x)+\tfrac{1}{6}(x_3-b(x))^3\\ &\qquad-\tfrac{1}{2}\big((1+\eta(x))^2-bx_3\big)(x_3-b(x))\big\}\Delta^2\phi(x)+O(\delta^6).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\text{(3.6)}\qquad \Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi=&-\nabla\cdot\big((1+\eta-b)\nabla\phi\big)-\delta^2\Delta\big(\tfrac{1}{3}(1+\eta-b)^3\Delta\phi\big)\\ &+\delta^2\Delta\big(\tfrac{1}{2}(1+\eta-b)^2\nabla b\cdot\nabla\phi\big)-\delta^2\nabla\cdot\big(\tfrac{1}{2}(1+\eta-b)^2(\nabla b)\Delta\phi\big)\\ &+\delta^2\nabla\cdot\big((1+\eta-b)(\nabla b)(\nabla b\cdot\nabla\phi)\big)+O(\delta^4).\end{aligned}$$

This formal expansion of the operator $\Lambda^{\mathrm{DN}}=\Lambda^{\mathrm{DN}}(\eta,b,\delta)$ with respect to δ^2 can be justified mathematically by the following lemma.

Lemma 3.1 ([1]) *Let $s>1$ and $M,c_1>0$. Suppose that*

$$\begin{cases}\|\eta\|_{s+9/2}+\|b\|_{s+11/2}\le M,\\ 1+\eta(x)-b(x)\ge c_1\quad\text{for}\quad x\in\mathbf{R}^2.\end{cases}$$

Then, there exists a constant $C=C(M,c_1,s)>0$ independent of δ such that for any $\delta\in(0,1]$ we have

$$\begin{aligned}&\big\|\Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi+\nabla\cdot\big((1+\eta-b)\nabla\phi\big)+\delta^2\Delta\big(\tfrac{1}{3}(1+\eta-b)^3\Delta\phi\big)\\ &-\delta^2\Delta\big(\tfrac{1}{2}(1+\eta-b)^2\nabla b\cdot\nabla\phi\big)+\delta^2\nabla\cdot\big(\tfrac{1}{2}(1+\eta-b)^2(\nabla b)\Delta\phi\big)\\ &-\delta^2\nabla\cdot\big((1+\eta-b)(\nabla b)\nabla b\cdot\nabla\phi\big)\big\|_s\le C\delta^4\|\nabla\phi\|_{s+11/2}.\end{aligned}$$

Similarly, we can obtain an expansion of the Neumann-to-Neumann map $\Lambda^{\mathrm{NN}}(\eta, b, \delta)$ with respect to δ^2, that is, letting Φ be the solution of the boundary value problem

$$\begin{cases} \Delta\Phi + \delta^{-2}\partial_3^2\Phi = 0 & \text{in } \Omega, \\ \Phi = 0 & \text{on } \Gamma, \\ \delta^{-2}\partial_3\Phi - \nabla b \cdot \nabla\Phi = \beta & \text{on } \Sigma, \end{cases}$$

we obtain

$$\nabla\Phi(x, x_3) = -\delta^2\beta(x)\nabla\eta(x) - \delta^2(1 + \eta(x) - x_3)\nabla\beta(x) + O(\delta^4)$$

and

$$\begin{aligned} \partial_3\Phi(x, x_3) &= \delta^2\beta(x) - \delta^4\nabla b(x) \cdot \big(\beta(x)\nabla\eta(x) + (1 + \eta(x) - b(x))\nabla\beta(x)\big) \\ &\quad + \delta^4(x_3 - b(x))\big(\nabla \cdot (\beta(x)\nabla\eta(x)) + \nabla\eta(x) \cdot \nabla\beta(x)\big) \\ &\quad - \tfrac{1}{2}\delta^4\big((1 + \eta(x) - x_3)^2 - (1 + \eta(x) - b(x))^2\big)\Delta\beta(x) + O(\delta^6). \end{aligned}$$

Hence, by the definition (2.10) with $\phi = 0$ we have

$$\Lambda^{\mathrm{NN}}(\eta, b, \delta)\beta = \beta + \delta^2\nabla \cdot \big((1 + \eta - b)(\nabla\eta)\beta + \tfrac{1}{2}(1 + \eta - b)^2\nabla\beta\big) + O(\delta^4). \tag{3.7}$$

This formal expansion of the operator $\Lambda^{\mathrm{NN}} = \Lambda^{\mathrm{NN}}(\eta, b, \delta)$ with respect to δ^2 can be justified mathematically by the following lemma.

Lemma 3.2 ([6]) *Let $s > 1$ and $M, c_1 > 0$. Suppose that*

$$\begin{cases} \|\eta\|_{s+9/2} + \|b\|_{s+11/2} \le M, \\ 1 + \eta(x) - b(x) \ge c_1 \quad \text{for} \quad x \in \mathbf{R}^2. \end{cases}$$

Then, there exist constants $C = C(M, c_1, s) > 0$ and $\delta_0 = \delta_0(M, c_1, s) > 0$ such that for any $\delta \in (0, \delta_0]$ we have

$$\big\|\Lambda^{\mathrm{NN}}(\eta, b, \delta)\beta - \beta - \delta^2\nabla \cdot \big\{(1 + \eta - b)\big(\beta\nabla\eta + \tfrac{1}{2}(1 + \eta - b)\nabla\beta\big)\big\}\big\|_s \le C\delta^4\|\beta\|_{s+4}.$$

It follows from (2.14), (3.5), and (3.7) that

$$\begin{cases} \eta_t + \nabla \cdot \big((1 + \eta - b)\nabla\phi\big) = b_t + O(\delta^2), \\ \phi_t + \eta + \tfrac{1}{2}|\nabla\phi|^2 = O(\delta^2), \end{cases}$$

which approximate the equations in (2.14) up to order $O(\delta^2)$. Now, putting $u = \nabla\phi$ and letting $\delta \to 0$ in the above equations we obtain

$$\begin{cases} \eta_t + \nabla \cdot \big((1 + \eta - b)u\big) = b_t, \\ u_t + (u \cdot \nabla)u + \nabla\eta = 0. \end{cases}$$

We proceed to derive higher order approximate equations. By (3.6) and (3.7), we can approximate the equations (2.14) by the following partial differential equations up to order $O(\delta^4)$.

$$
(3.8)\qquad \begin{cases} \eta_t - b_t + \nabla\cdot\big((1+\eta-b)\nabla\phi\big) + \delta^2\Delta\big(\tfrac{1}{3}(1+\eta-b)^3\Delta\phi\big) \\ \quad -\delta^2\Delta\big(\tfrac{1}{2}(1+\eta-b)^2\nabla b\cdot\nabla\phi\big) + \delta^2\nabla\cdot\big(\tfrac{1}{2}(1+\eta-b)^2\nabla b\Delta\phi\big) \\ \quad -\delta^2\nabla\cdot\big((1+\eta-b)\nabla b(\nabla b\cdot\nabla\phi)\big) \\ \quad -\delta^2\nabla\cdot\big\{(1+\eta-b)\big(b_t\nabla\eta + \tfrac{1}{2}(1+\eta-b)\nabla b_t\big)\big\} = O(\delta^4), \\ \phi_t + \eta + \tfrac{1}{2}|\nabla\phi|^2 - \tfrac{1}{2}\delta^2\big(\nabla b\cdot\nabla\phi - (1+\eta-b)\Delta\phi + b_t\big)^2 = O(\delta^4). \end{cases}
$$

Here, we define a second order partial differential operator $T(\eta, b)$ depending on η and b and acting on vector fields by

$$
\begin{aligned} T(\eta,b)u := & -\nabla\big(\tfrac{1}{3}(1+\eta-b)^3(\nabla\cdot u)\big) + \nabla\big(\tfrac{1}{2}(1+\eta-b)^2(\nabla b\cdot u)\big) \\ & -\tfrac{1}{2}(1+\eta-b)^2\nabla b(\nabla\cdot u) + (1+\eta-b)\nabla b(\nabla b\cdot u) \end{aligned}
$$

and introduce a new variable u by

$$
(3.9)\qquad \nabla\phi = u + \delta^2(1+\eta-b)^{-1}T(\eta,b)u + \delta^2\big(b_t\nabla\eta + \tfrac{1}{2}(1+\eta-b)\nabla b_t\big).
$$

Putting this into equations (3.8) and neglecting the terms of order $O(\delta^4)$, we obtain the Green–Naghdi equation

$$
(3.10)\qquad \begin{cases} \eta_t + \nabla\cdot\big((1+\eta-b)u\big) = b_t, \\ \big((1+\eta-b) + \delta^2 T(\eta,b)\big)u_t + (1+\eta-b)\big(\nabla\eta + (u\cdot\nabla)u\big) \\ \quad +\delta^2\big\{\tfrac{1}{3}\nabla\big((1+\eta-b)^3 P_u(\nabla\cdot u)\big) + Q(\eta,u,b) \\ \qquad +R_1(\eta,u,b)b_t + R_2(\eta,b)b_{tt}\big\} = 0 \quad \text{for} \quad t>0, \end{cases}
$$

$$
(3.11)\qquad \eta = \eta_0, \quad u = u_0 \quad \text{at} \quad t = 0,
$$

where

$$
\begin{aligned} & P_u = \nabla\cdot u - u\cdot\nabla, \\ & Q(\eta,u,b) = \tfrac{1}{2}\nabla\big((1+\eta-b)^2(u\cdot\nabla)^2 b\big) + \tfrac{1}{2}\big((1+\eta-b)^2 P_u(\nabla\cdot u)\big)\nabla b \\ & \qquad +(1+\eta-b)\big((u\cdot\nabla)^2 b\big)\nabla b, \\ & R_1(\eta,u,b)b_t = (1+\eta-b)^2\nabla(u\cdot\nabla b_t) + 2(1+\eta-b)(u\cdot\nabla b_t)\nabla\eta, \\ & R_2(\eta,b)b_{tt} = \tfrac{1}{2}(1+\eta-b)^2\nabla b_{tt} + (1+\eta-b)b_{tt}\nabla\eta, \end{aligned}
$$

and u_0 is determined by (3.9) from (η_0, b_0). Now, we are ready to give the main result in this paper, which asserts the rigorous justification of the Green–Naghdi approximation.

Theorem 3.1 *Let $s > 3$ and $M_0, c_0 > 0$. Then, there exist a time $T > 0$ and constants $C, \delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, $\eta_0 \in H^{s+15/2}$, $\nabla\phi_0 \in H^{s+7}$, and $b \in C([0,T]; H^{s+8})$ satisfying*

$$\begin{cases} \|b(t)\|_{s+8} + \|b_t(t)\|_{s+7} + \|b_{tt}(t)\|_{s+5} + \|b_{ttt}(t)\|_{s+4} \le M_0, \\ \|\eta_0\|_{s+15/2} + \|\nabla\phi_0\|_{s+7} \le M_0, \\ 1 + \eta_0(x) - b_0(x) \ge c_0 \quad \text{for} \quad (x,t) \in \mathbf{R}^2 \times [0,T], \end{cases}$$

the solution $(\eta, \phi) = (\eta^\delta, \phi^\delta)$ obtained in Theorem 2.1 and the function u^δ determined by (3.9) from $(\eta^\delta, \phi^\delta)$ and b satisfy

$$\|\eta^\delta(t) - \tilde{\eta}^\delta(t)\|_s + \|u^\delta(t) - \tilde{u}^\delta(t)\|_s + \delta\|\nabla \cdot (u^\delta(t) - \tilde{u}^\delta(t))\|_s \le C\delta^4$$

for $0 \le t \le T$, where $(\eta, u) = (\tilde{\eta}^\delta, \tilde{u}^\delta)$ is a unique solution of the initial value problem for the Green–Naghdi equation (3.10) and (3.11).

4 The Green–Naghdi equations

We first explain what are the Green–Nagdhi equations and why we introduce the new variable u by the formula (3.9). For simplicity, we consider a linearized problem around the trivial flow in the case of a flat bottom. Since the Dirichlet-to-Neumann map in the trivial case can be written explicitly in terms of the Fourier multipliers as $\Lambda^{\rm DN}(0,0,\delta) = \frac{1}{\delta}|D|\tanh(\delta|D|)$, the linearized equations for the full equations (2.14) have the form

$$\begin{cases} \phi_t + \eta = 0, \\ \eta_t - \frac{1}{\delta}|D|\tanh(\delta|D|)\phi = 0. \end{cases} \tag{4.1}$$

Using the Taylor expansion $\tanh x = x + O(x^3)$ $(x \to 0)$, we have

$$\tfrac{1}{\delta}|D|\tanh(\delta|D|) = |D|^2 + O(\delta^2) = -\Delta + O(\delta^2),$$

so that the linearized equations (4.1) can be approximated by the partial differential equations up to order $O(\delta^2)$ as

$$\begin{cases} \eta_t + \Delta\phi = O(\delta^2), \\ \phi_t + \eta = 0. \end{cases}$$

Letting $\delta \to 0$ we obtain linearized shallow water equations.

To obtain a higher order approximation, we use the Taylor expansion $\tanh x = x - \frac{1}{3}x^3 + O(x^5)$ $(x \to 0)$. Then, we have

$$\begin{aligned} \tfrac{1}{\delta}|D|\tanh(\delta|D|) &= |D|^2 - \tfrac{1}{3}\delta^2|D|^4 + O(\delta^4) \\ &= -\Delta - \tfrac{1}{3}\delta^2\Delta^2 + O(\delta^4). \end{aligned}$$

Putting this into the linearized equations (4.1) and neglecting the terms of order $O(\delta^4)$, we obtain higher order approximate equations

$$
\text{(4.2)} \qquad \begin{cases} \eta_t + \Delta\phi + \frac{1}{3}\delta^2\Delta^2\phi = 0, \\ \phi_t + \eta = 0. \end{cases}
$$

This system has a non-trivial solution of the form

$$
\eta(x,t) = \eta_0 \mathrm{e}^{\mathrm{i}(\xi\cdot x - \omega t)}, \quad \phi(x,t) = \phi_0 \mathrm{e}^{\mathrm{i}(\xi\cdot x - \omega t)}
$$

if the wave vecotr $\xi \in \mathbf{R}^2$ and the angler frequency $\omega \in \mathbf{C}$ satisfy

$$
\omega^2 - \left(1 - \tfrac{1}{3}\delta^2|\xi|^2\right)|\xi|^2 = 0,
$$

which is the so-called dispersion relation for (4.2). In the case $|\xi| > \frac{3}{\delta^2}$, the solutions ω of this dispersion relation are purely imaginary and given by $\omega = \pm \mathrm{i}|\xi|\sqrt{\frac{1}{3}\delta^2|\xi|^2 - 1}$, so that the approximate equations (4.2) have a solution of the form

$$
\eta(x,t) = \eta_0 \mathrm{e}^{\mathrm{i}\xi\cdot x + t|\xi|\sqrt{\frac{1}{3}\delta^2|\xi|^2 - 1}},
$$

which grows exponentially as $|\xi| \to \infty$ for each $t > 0$. Therefore, the initial value problem for (4.2) is in general ill-posed, and (4.2) is not good approximation for the linearized equations (4.1).

On the other hand, in view of the relation

$$
\frac{|D|\tanh(\delta|D|)}{\delta}\phi = -\Delta\left(\phi + \tfrac{1}{3}\delta^2\Delta\phi\right) + O(\delta^4),
$$

let us introduce a new variable ψ satisfying the relation

$$
\phi + \tfrac{1}{3}\delta^2\Delta\phi = \psi + O(\delta^4).
$$

This implies that $\phi = \psi + O(\delta^2)$, so that

$$
\phi = \psi - \tfrac{1}{3}\delta^2\Delta\phi + O(\delta^4) = (1 - \tfrac{1}{3}\delta^2\Delta)\psi + O(\delta^4).
$$

This motivates us to introduce a new variable ψ by

$$
\psi = \left(1 - \tfrac{1}{3}\delta^2\Delta\right)^{-1}\phi.
$$

Then, it follows from (4.1) that

$$
\begin{cases} \eta_t + \Delta\psi = O(\delta^4), \\ \psi_t + \eta = \frac{1}{3}\delta^2\Delta\psi_t. \end{cases}
$$

Putting $u = \nabla\psi$ and neglecting the term of order $O(\delta^4)$, we obtain

$$(4.3) \qquad \begin{cases} \eta_t + \nabla\cdot u = 0, \\ u_t + \nabla\eta = \frac{1}{3}\delta^2\Delta u_t. \end{cases}$$

We note that if we use the Padé approximation

$$\tanh x = \frac{x}{1+\frac{1}{3}x^2} + O(x^5) \quad (x \to 0)$$

in place of the Taylor expansion $\tanh x = x - \frac{1}{3}x^3 + O(x^5)$, we can directly obtain the linearized Green–Nagdhi equations (4.3) from the linearized water wave equations (4.1). The dispersion relation for (4.3) is

$$(1 + \tfrac{1}{3}\delta^2|\xi|^2)\omega^2 - |\xi|^2 = 0,$$

so that the initial value problem for (4.3) is well-posed. In fact, for any smooth solution for (4.3) we have the following energy equality.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\eta(t)\|_s^2 + \|u(t)\|_s^2 + \tfrac{1}{3}\delta^2\|\nabla\cdot u(t)\|_s^2\Big\} = 0.$$

Therefore, we can expect that the solution for (4.1) can be approximated by the solution (4.3) up to order $O(\delta^4)$. Corresponding nonlinear equations are the Green–Nagdhi equations.

Next, we consider the initial value problem for the Green–Nagdhi equations (3.10) and (3.11). We first show that the change of variables by (3.9) is well-defined. To this end, we define a second order differential operator $L(\eta, b, \delta)$ by

$$L(\eta, b, \delta)u := \big((1+\eta-b) + \delta^2 T(\eta, b)\big)u$$

and consider the partial differential equation

$$(4.4) \qquad L(\eta, b, \delta)u = F + \delta a\nabla f.$$

It is easy to see that

$$\begin{aligned}(Lu, \phi) = ((1+\eta-b)u, \phi) &+ \tfrac{\delta^2}{3}((1+\eta-b)^3(\nabla\cdot u), \nabla\cdot\phi) \\ &- \tfrac{\delta^2}{2}((1+\eta-b)^2(\nabla b\cdot u), \nabla\cdot\phi) - \tfrac{\delta^2}{2}((1+\eta-b)^2(\nabla\cdot u), \nabla b\cdot\phi) \\ &+ \delta^2((1+\eta-b)(\nabla b\cdot u), \nabla b\cdot\phi).\end{aligned}$$

Therefore, under appropriate assumptions on η and b we have

$$(4.5) \qquad C^{-1}(\|u\|^2 + \delta^2\|\nabla\cdot u\|^2) \le (Lu, u) \le C(\|u\|^2 + \delta^2\|\nabla\cdot u\|^2).$$

Thus, we can show the existence of the solution to (4.4) satisfying the estimate $\|u\| + \delta\|\nabla\cdot u\| \le C(\|F\| + \|f\|)$. More precisely, we have the following lemma.

Lemma 4.1 *Let $s > 2$ and $M, c_1 > 0$. Suppose that*

$$\begin{cases} \|\eta\|_s + \|b\|_{s+1} + \|a\|_s \leq M, \\ 1 + \eta(x) - b(x) \geq c_1 \quad \text{for} \quad x \in \mathbf{R}^2. \end{cases}$$

Then, for any $F, f \in H^s$ and $\delta \in (0,1]$, equation (4.4) *has a unique solution $u \in H^s$ satisfying $\nabla \cdot u \in H^s$. Moreover, we have*

$$\|u\|_s + \delta\|\nabla \cdot u\|_s \leq C(\|F\|_s + \|f\|_s),$$

where $C = C(M, c_1, s) > 0$ is independent of δ.

Next, we consider linearized equations around a flow (η, u) and give an energy estimate of the solution to the linearized equations. Letting $\zeta := \partial\eta$ and $w := \partial u$ we can write the linearized equations as

$$(4.6) \qquad \begin{cases} \zeta_t + \nabla \cdot (hw) + \nabla \cdot (\zeta u) = f_1, \\ Lw_t + h(\nabla\zeta + (u \cdot \nabla)w) \\ \quad -\frac{1}{3}\delta^2\nabla(h^3(u \cdot \nabla)(\nabla \cdot w)) + \delta^2 h(\nabla b)(u \cdot \nabla)(w \cdot \nabla b) \\ \quad +\frac{1}{2}\delta^2\nabla(h^2(u \cdot \nabla)(w \cdot \nabla b)) - \frac{1}{2}\delta^2(\nabla b)\nabla \cdot (h^2 u(\nabla \cdot w)) \\ \quad +\delta^2\nabla(a_1\zeta) + a_2\zeta + \delta^2\nabla(a_3(\nabla \cdot w) + a_4 \cdot w) + \delta^2 a_5(\nabla \cdot w) + A_6 w \\ \quad = f_2 + \delta\nabla f_3, \end{cases}$$

where $h = 1 + \eta - b$. $a_1 = a_1(\eta, b, u)$ and $a_3 = a_3(\eta, b, u)$ are scalar valued functions, $a_2 = a_2(\eta, b, u)$, $a_4 = a_4(\eta, b, u)$, and $a_5 = a_5(\eta, b, u)$ are vector valued functions, and $A_6 = A_6(\eta, b, u)$ is a matrix valued function. We can write down explicitly these functions in terms of η, b, and u. However, we omit it since the explicit forms are not important in our purpose. The basic energy function for these linearized equations is defined by

$$\mathcal{E}(t) := \|\zeta(t)\|^2 + (L(\eta, b, \delta)w, w).$$

In view of (4.5), under appropriate assumptions on η and b, this energy function is equivalent to

$$E(t) := \|\zeta(t)\|^2 + \|w(t)\|^2 + \delta^2\|\nabla \cdot w(t)\|^2$$

uniformly with respect to $\delta \in (0,1]$. For any smooth solution (ζ, w) to the linearized equations (4.6), we see that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) &= (\zeta,\zeta_t) + (w, Lw_t) + \tfrac{1}{2}(w,[\partial_t,L]w) \\
&= -(\zeta,\nabla\cdot(hw)) - (\zeta,\nabla\cdot(\zeta u)) + (\zeta,f_1) - (w,h\nabla\zeta) - (w,h(u\cdot\nabla)w) \\
&\quad + \tfrac{1}{3}\delta^2(w,\nabla(h^3(u\cdot\nabla)(\nabla\cdot w))) - \delta^2(w,h(\nabla b)(u\cdot\nabla)(w\cdot\nabla b)) \\
&\quad - \tfrac{1}{2}\delta^2(w,\nabla(h^2(u\cdot\nabla)(w\cdot\nabla b)) - (\nabla b)\nabla\cdot(h^2u(\nabla\cdot w))) \\
&\quad - \delta^2(w,\nabla(a_1\zeta)) - (w,a_2\zeta) - \delta^2(w,\nabla(a_3(\nabla\cdot w)+a_4\cdot w)) \\
&\quad - \delta^2(w,a_5(\nabla\cdot w)) - (w,A_6w) + (w,f_2+\delta\nabla f_3) + \tfrac{1}{2}(w,[\partial_t,L]w).
\end{aligned}$$

Here, we have

$$\begin{aligned}
&(\zeta,\nabla\cdot(hw)) + (w,h\nabla\zeta) = 0, \\
&(\zeta,\nabla\cdot(\zeta u)) = \tfrac{1}{2}(\zeta,(\nabla\cdot u)\zeta), \\
&(w,h(u\cdot\nabla)w) = -\tfrac{1}{2}(w,(\nabla\cdot(hu))w), \\
&(w,\nabla(h^3(u\cdot\nabla)(\nabla\cdot w))) = \tfrac{1}{2}(\nabla\cdot w,(\nabla\cdot(h^3u))\nabla\cdot w), \\
&(w,h(\nabla b)(u\cdot\nabla)(w\cdot\nabla b)) = -\tfrac{1}{2}(w\cdot\nabla b,(\nabla\cdot(hu))w\cdot\nabla b), \\
&(w,\nabla(h^2(u\cdot\nabla)(w\cdot\nabla b)) - (\nabla b)\nabla\cdot(h^2u(\nabla\cdot w))) = 0,
\end{aligned}$$

so that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) &= -\tfrac{1}{2}(\zeta,(\nabla\cdot u)\zeta) + (\zeta,f_1) + \tfrac{1}{2}(w,(\nabla\cdot(hu))w) \\
&\quad + \tfrac{1}{6}(\nabla\cdot w,(\nabla\cdot(h^3u))\nabla\cdot w) + \tfrac{1}{2}(w\cdot\nabla b,(\nabla\cdot(hu))w\cdot\nabla b) \\
&\quad + \delta^2(\nabla\cdot w,a_1\zeta) - (w,a_2\zeta) + \delta^2(\nabla\cdot w,a_3(\nabla\cdot w)+a_4\cdot w) \\
&\quad - \delta^2(w,a_5(\nabla\cdot w)) - (w,A_6w) + (w,f_2) - \delta(\nabla\cdot w,f_3) + \tfrac{1}{2}(w,[\partial_t,L]w) \\
&\le C(\|\zeta(t)\|^2 + \|w(t)\|^2 + \delta^2\|\nabla\cdot w(t)\|^2) + \|f_1(t)\|^2 + \|f_2(t)\|^2 + \|f_3(t)\|^2 \\
&\le C\mathcal{E}(t) + \|f_1(t)\|^2 + \|f_2(t)\|^2 + \|f_3(t)\|^2.
\end{aligned}$$

Therefore, Gronwall's inequality gives

$$E(t) \le C\mathrm{e}^{Ct}\Big(E(0) + \int_0^t (\|f_1(\tau)\|^2 + \|f_2(\tau)\|^2 + \|f_3(\tau)\|^2)\mathrm{d}\tau\Big).$$

A higher order energy function is defined by

$$\mathcal{E}_s(t) := \|\eta(t)\|_s^2 + (L(1+|D|)^s u, (1+|D|)^s u).$$

Under appropriate assumptions on η and b, this energy function is equivalent to $E_s(t) := \|\eta(t)\|_s^2 + \|u(t)\|_s^2 + \delta^2\|\nabla\cdot u(t)\|_s^2$ uniformly with respect to $\delta \in (0,1]$. Similar calculation as above yields the energy estimate

$$E_s(t) \le C\mathrm{e}^{Ct}\Big(E_s(0) + \int_0^t (\|f_1(\tau)\|_s^2 + \|f_2(\tau)\|_s^2 + \|f_3(\tau)\|_s^2)\mathrm{d}\tau\Big).$$

To construct the solution, we use, for example, a parabolic regularization of the equations by

$$
(4.7)\qquad \begin{cases} \eta_t - \varepsilon\Delta\eta + \nabla\cdot\big((1+\eta-b)u\big) = b_t, \\ \big((1+\eta-b)+\delta^2 T(\eta,b)\big)(u_t - \varepsilon\Delta u) \\ \qquad +(1+\eta-b)\big(\nabla\eta + (u\cdot\nabla)u\big) \\ \qquad +\delta^2\{\frac{1}{3}\nabla\big((1+\eta-b)^3 P_u(\nabla\cdot u)\big) + Q(\eta,u,b) \\ \qquad\qquad +R_1(\eta,u,b)b_t + R_2(\eta,b)b_{tt}\} = 0 \quad \text{for} \quad t>0. \end{cases}
$$

For each $\varepsilon \in (0,1]$ the initial value problem for the regularized Green–Nagdhi equation (4.7) and (3.11) has a unique solution $(\eta^\varepsilon, u^\varepsilon)$, which satisfies a uniform bound on a time interval independent of ε. Moreover, the solution $(\eta^\varepsilon, u^\varepsilon)$ converges as $\varepsilon \to +0$. The limiting function is the desired solution. More precisely, we have the following proposition which asserts the existence of the solution to the initial value problem (3.10) and (3.11) with a uniform bound of the solution on a time interval independent of $\delta \in (0,1]$.

Proposition 4.1 *Let $s>3$ and $M, c_1 > 0$. Then, there exist a time $T>0$ and a constant $C_0 >$ such that for any $\delta \in (0,1]$, $\eta_0 \in H^s$, $u_0 \in H^s$, and $b \in C([0,T]; H^{s+2})$ satisfying*

$$
\begin{cases} \|\eta_0\|_s + \|u_0\|_s + \delta\|\nabla\cdot u_0\|_s \le M, \\ \|b(t)\|_{s+2} + \|b_t(t)\|_{s+2} + \|b_{tt}(t)\|_{s+1} \le M, \\ 1+\eta_0(x) - b_0(x) \ge c_1 \quad \text{for} \quad (x,t) \in \mathbf{R}^2 \times [0,T], \end{cases}
$$

the initial value problem for the Green–Naghdi equation (3.10) *and* (3.11) *has a unique solution (η, u) on the time interval $[0,T]$ satisfying*

$$
\begin{cases} \|\eta(t)\|_s + \|u(t)\|_s + \delta\|\nabla\cdot u(t)\|_s \le C_0, \\ 1+\eta(x,t) - b(x,t) \ge c_0/2 \quad \text{for} \quad (x,t) \in \mathbf{R}^2 \times [0,T]. \end{cases}
$$

5 Proof of the main theorem

Let $(\eta^\delta, \phi^\delta)$ be the solution of the full water wave problem (2.14) and (2.15) obtained in Theorem 2.1 and define u^δ by

$$
L(\eta^\delta, b, \delta)u^\delta = (1+\eta^\delta - b)\big(\nabla\phi^\delta - \delta^2(b_t\nabla\eta^\delta + \tfrac{1}{2}(1+\eta^\delta-b)\nabla b_t)\big).
$$

Then, we have

$$
(5.1)\qquad \begin{cases} \eta_t^\delta + \nabla\cdot\big((1+\eta^\delta-b)u^\delta\big) = b_t + \delta^4 g_1^\delta, \\ L(\eta^\delta,b,\delta)u_t^\delta + (1+\eta^\delta-b)\big(\nabla\eta^\delta + (1+\eta^\delta-b)(u^\delta\cdot\nabla)u^\delta\big) \\ \qquad +\delta^2\{\frac{1}{3}\nabla\big((1+\eta^\delta-b)^3 P_{u^\delta}(\nabla\cdot u^\delta)\big) + Q(\eta^\delta,u^\delta,b) \\ \qquad\qquad +R_1(\eta^\delta,u^\delta,b)b_t + R_2(\eta^\delta,b)b_{tt}\} = \delta^4 g_2^\delta, \end{cases}
$$

where g_1^δ and g_2^δ are uniformly bounded with respect to $\delta \in (0,1]$. In fact, we have the following lemma.

Lemma 5.1 *Under the same hypothesis of Theorem* 3.1, *there exists a constant* $C = C(M_0, c_0, s) > 0$ *such that we have*

$$\|(g_1^\delta(t), g_2^\delta(t))\|_s \le C \quad \text{for} \quad t \in [0,T], \quad \delta \in (0, \delta_0],$$

where T *and* δ_0 *are the constants in Theorem* 2.1.

Let $(\tilde{\eta}^\delta, \tilde{u}^\delta)$ be the solution of the Green–Nagdhi equations (3.10) and (3.11) obtained in Proposition 4.1 and put

$$\zeta = \eta^\delta - \tilde{\eta}^\delta, \quad w = u^\delta - \tilde{u}^\delta.$$

Then, we see that ζ and w satisfy linearized Green–Nagdhi equations (4.6) with appropriately modified coefficients and $(f_1, f_2, f_3) = \delta^4(g_1, g_2, 0)$. We also have $(\zeta, w) = 0$ at $t = 0$. Therefore, we obtain

$$E_s(t) \le C\delta^8 \int_0^t \mathrm{e}^{C(t-\tau)}(\|g_1(\tau)\|_s^2 + \|g_2(\tau)\|_s^2)\mathrm{d}\tau \le C\delta^8,$$

which implies the desired estimate

$$\|\zeta(t)\|_s + \|w(t)\|_s + \delta\|\nabla \cdot w(t)\|_s \le C\delta^4.$$

The details will be published elsewhere.

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